Finite-size scaling and universality for the totally asymmetric simple-exclusion process

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The applicability of the concepts of finite-size scaling and universality to nonequilibrium phase transitions is considered in the framework of the one-dimensional totally asymmetric simple-exclusion process with open boundaries. In the thermodynamic limit there are boundary-induced transitions both of the first and second order between steady-state phases of the model. We derive finite-size scaling expressions for the current near the continuous phase transition and for the local density near the first-order transition under different stochastic dynamics and compare them to establish the existence of universal functions. Next we study numerically the finite-size behavior of the Lee-Yang zeros of the normalization factor for the different steady-state probabilities.

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I. INTRODUCTION

Usually, the most general and fundamental properties of equilibrium phase transitions are formulated as scaling laws for thermodynamic and correlation functions. The scaling hypothesis states that certain such functions become (generalized) homogeneous functions of the relevant variables close to the critical point. According to the universality hypothesis the large variety of different models can be divided into a few classes with respect to their critical behavior, depending on the dimensionality of the system, the symmetry of the order parameter, and the range of interaction (finite or infinite). Universality implies that the scaling functions are the same for all systems within a given universality class, as much as the critical exponents are.

Since the beginning of the 1970s, when the basic ideas of finite-size scaling (FSS) were stated by Fisher [1] and Fisher and Barber [2], a comprehensive theory has been developed; see the reviews in [3,4] and the recent book in [5]. Close to the bulk critical point, when the bulk correlation length becomes comparable to the characteristic size of the system, this theory offers a description in terms of finite-size scaling functions, the universality of which depends, in addition, on the shape of the system and the type of the boundary conditions. The asymptotic behavior of these functions reveals an intimate mechanism of how the singularities build up near a phase transition point as the thermodynamic limit is approached.

A quite different program of analysis of the appearence of singularities in the thermodynamic limit has been offered in the classical paper by Yang and Lee [6]. They studied how the zeros of the Ising model partition function in the complex magnetic field plane approach the real axis with the increase of the number of spins. Fisher [7] extended their approach to the zeros of the partition function in the complex temperature plane. By calculating in the thermodynamic limit the line of zeros and their density near the positive real axis one can exactly locate the transition point and obtain the order of the phase transition.

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It seems quite challenging to extend the above ideas to the case of phase transitions far from equilibrium. In general, for stochastic processes it is not clear how to define a meaningful energy function on the configuration space that would be consistent with both the transition probabilities (or rates) and the stationary probability distribution. The problem of finding analogs of the equilibrium partition function and thermodynamic potential for models which do not satisfy the detailed balance condition is still open. The first exact results in that direction have been obtained recently by Derrida, Lebowitz, and Speer for the symmetric [8] and asymmetric [9] simple-exclusion process.

To our knowledge, finite-size scaling at a nonequilibrium critical point and its universality have been checked for the pair-contact process with and without diffusion; for a review, see [10]. For the asymmetric simple-exclusion process (ASEP), finite-size scaling at both the continuous and first-order transitions has been established only in the case of forward-ordered sequential dynamics by one of the present authors [11]. It is one of the aims of this work to check if the same form of the FSS functions holds for the ASEP with continuous-time dynamics and with a differnt type of discrete-time dynamics—the parallel one.

It is worth mentioning that the model and its generalizations have found numerous applications for the description of such diverse phenomena as kinetics of biopolymerization [12,13], single-lane vehicular traffic [14–16], data packet transport in the Internet [17], surface growth [18], shock structures [19,20], directed movement of molecular motors along filaments [21,22], and statistical significance of sequence alignments [23]; see also the reviews in [24,25].

Essential progress has been made recently in the extension of the Lee-Yang theory to special nonequilibrium cases. Arndt [26] recognized that the normalization factor of the stationary probability distribution for a one-dimensional stochastic diffusion model with two types of particles plays the role of a grand canonical partition function. By considering its roots in the complex fugacity plane for finite numbers of sites, he was able to distinguish two regions of analiticity which were identified with two different phases. The constant density of roots at the positive real axis in the thermodynamic limit is an evidence for a first-order phase transi-

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tion. These observations have initiated an avalanche of similar investigations in the context of various models exhibiting nonequilibrium phase transitions. Arndt *et al.* [27] and Dammer *et al.* [28] studied the zeros of the survival probability for models in the directed percolation (DP) universality class. As is well known, DP can be interpreted as a dynamical process by considering the direction of activity spreading as temporal. It was noted [28] that the distribution of the zeros in the plane of complex probability of an open bond provides information about universal properties, such as the critical exponents of the spatial and temporal correlation lengths. Next, Blythe and Evans [29] found that the distribution of the zeros of the stationary probability normalization factor for the continuous-time ASEP with open boundaries, considered in the plane of a complex injection rate, agrees with the Lee-Yang equilibrium theory for both the first- and second-order phase transitions. A review of some recent work in which the Lee-Yang theory has been successfully applied to nonequilibrium phase transitions is given in [30]. Next, Bena *et al.* [31] showed that the Lee-Yang strategy works also for a mean-field-like urn model for sand separation. Moreover, they observed that the rate of convergence of the zeros of the probability normalization factor to the transition point obeys the FSS prediction for the finite-size shift of the pseudocritical temperature. Jafarpour found further evidence for the applicability of the Lee-Yang theory to a model of two types of particles hopping in opposite directions on a ring [32] and to a reaction-diffusion model on a chain with open boundary [33].

The real breakthrough in our understanding of the applicability of the Lee-Yang theory to ASEP occurred this year. Blythe *et al.* [34,35] have found mappings of the normalization factor for the parallel-update ASEP onto several thermodynamically equivalent two-dimensional lattice path problems involving weighted Dyck or Motzkin paths. They have shown [35] that the critical behavior of the ASEP is a combination of that for two adsorbing Dyck walks: the second-order phase transition corresponds to adsorption or desorption of one of the walks and the first-order one to a cooperative transition of the two chains. Brak and Essam [36] have shown that the three matrix representations of the algebra of the continuous-time ASEP (the so-called DEHP algebra) can be interpreted as transfer matrices for different weighted lattice path problems. Brak, de Gier, and Rittenberg [37] constructed a new representation of the DEHP algebra which gives the transfer matrix for a one-transit-walk model. The totally ASEP (TASEP) current and density were related to the equilibrium densities of the latter model. Furthermore, the transition between the disordered low- or high-density phases and the maximum-current phase was interpreted as a special surface phase transition with a single critical exponent $\phi=1/2$, known in polymer physics. The above results allow one to regard the normalization factor of the ASEP stationary probability distribution as an equilibrium configuration sum for certain polymer chains interacting with a fixed interface.

The paper is organised as follows. A brief description of the finite-size TASEP with open boundaries and different types of dynamics is given in Sec. II. Here the known relationships between the corresponding exact finite-size expressions for the normalization factor of the stationary probability distribution, the stationary current, and local density are given too. They are analyzed within the framework of finite-size scaling for the current at the second-order phase transition in Sec. III and for the particle density at the first-order phase transition in Sec. IV. In each case the FSS functions corresponding to the different updates are compared and their universal shape is established. The distribution of the zeros of the normalization factor for the TASEP with discrete-time updates is investigated in Sec. V and compared to the known results for continuous-time dynamics. The rate of convergence of the zeros to the critical point, as the number of sites in the chain increases, is estimated numerically. The paper closes with Sec. VI containing the general conclusions.

II. MODEL

We consider the totally asymmetric simple-exclusion process on a finite chain of L sites with open boundaries. For a mathematical definition of the exclusion processes we refer the reader to the book in [38], and for a recent review on exactly solvable models far from equilibrium to [25]. The configurations of the system are defined by a set of L binary variables $\{\tau_1, \tau_2, \dots, \tau_L\}$, where $\tau_i=0$ ($\tau_i=1$) means that site i of the chain is empty (occupied by exactly one particle). The particles obey a stochastic dynamics according to which they may hop only to empty nearest-neighbor sites to the right. The open boundary conditions imply that a particle can be injected at the left end of the chain ($i=1$) and removed at the right end ($i=L$). The order in which the local hopping, injection and particle removal take place in space and time defines the dynamics—i.e., the way in which the configurations are updated in the course of time. Basically, one distinguishes between continuous-time and discrete-time updates. A realisation of the continuous-time master equation is given by the random-sequential update. In this case during every infinitesimal time step $t \to t+\mathrm{d}t$ each particle on site $i \in \{1, 2, \dots, L-1\}$ can hop to the right with probability $p\mathrm{d}t$ provided the target site is free. In addition, a particle is injected at site $i=1$ with probability $\alpha\mathrm{d}t$ if $\tau_1=0$ and particle is removed from site $i=L$ with probability $\beta\mathrm{d}t$ if $\tau_L=1$. Otherwise the configuration remains unchanged. By rescaling of time one can always set $p=1$. In the case of discrete-time updates, during the time step $t \to t+1$ attempts are made to change the state of each lattice site: hopping to the right with probability p, injection at $i=1$, and removal from $i=L$ with probabilities α and β, respectively. According to the order of performance of these attempts, there are four basic cases: forward- ($\rightarrow$) and backward- ($\leftarrow$) ordered sequential, sublattice parallel (s-$\|$), and fully parallel ($\|$). Their precise definition is given, e.g., in [39]. For easy reference they are distinguished here by the corresponding superscripts $\rightarrow$, $\leftarrow$, s-$\|$, and $\|$.

In all of the above cases it has been proved that the steady-state probability $P(\tau_1, \tau_2, \dots, \tau_L)$ can be written in the form of a scalar product of noncommuting matrices (the so-called matrix-product ansatz):

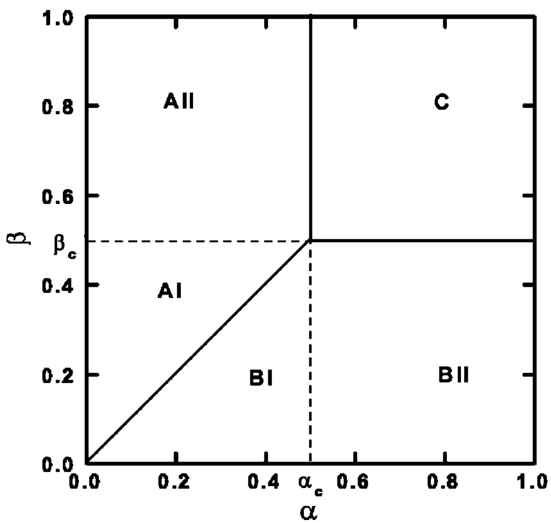

FIG. 1. The phase diagram in the plane of the injection and removal probabilities α and β (see the text) for hopping probability $p=0.75$. The maximum-current phase occupies region C. Region $A=AI\cup AII$ corresponds to the low-density phase and region $B=BI\cup BII$ to the high-density phase. Subregions AI (BI) and AII (BII) are distinguished by the different analytic form of the density profile. The boundary between them, $\beta=\beta_c$, $0\leqslant\alpha\leqslant\alpha_c$ ($\alpha=\alpha_c$, $0\leqslant\beta\leqslant\beta_c$), is shown by the dashed segment of a straight line. The solid line $\alpha=\beta$ between subregions AI and BI is the coexistence line of the low- and high-density phases.

$$P(\tau_1,\tau_2,\ldots,\tau_L)=Z_L^{-1}\langle W|\prod_{i=1}^{L}[\tau_i D+(1-\tau_i)E]|V\rangle. \quad (1)$$

Here the two matrices D and E act on the vectors of an auxiliary (in general infinite-dimensional) vector space $\mathcal{S}$, the vectors $|V\rangle\in\mathcal{S}$ and $\langle W|\in\mathcal{S}^\dagger$, where $\mathcal{S}^\dagger$ is the dual of $\mathcal{S}$. The normalization factor Z_L is of special interest for us and will be considered in detail in Sec. V.

We mention that the case of random-sequential update was solved first by using the recursion relations method [40,41] and then by means of the matrix-product ansatz (MPA) [42]. Next, the method of the MPA was successfully applied for obtaining the steady-state properties in all the basic cases of discrete-time dynamics: forward- and backward-ordered sequential [43–45], sublattice parallel [46,47], and fully parallel dynamics [48,49].

The phase diagram for all the discrete-time updates has the same structure as shown in Fig. 1: it contains maximum-current (MC), low-density (LD), and high-density (HD) phases. The maximum-current phase is separated by lines of continuous phase transitions ($\alpha=\alpha_c,\beta_c\leqslant\beta\leqslant 1$) and ($\beta=\beta_c,\alpha_c\leqslant\alpha\leqslant 1$) from the low-density and high-density phases, respectively. Here α_c and β_c are the critical values of the injection and removal probabilities:

$$\alpha_c=\beta_c=1-\sqrt{1-p}. \quad (2)$$

The above phases are identified with respect to the analytic form of the bulk current: for fixed p, the current in the low- (high-) density phase depends only on α (β), and in the maximum-current phase it is independent of both α and β. On crossing the borderline between the maximum-current and the low- (high-) density phase, the current itself and its first derivative with respect to α (β) change continously, and the second derivative with respect to α (β) udergoes a finite jump. The coexistence line between the low- and high-density phases is given by $\alpha=\beta$, $0\leqslant\beta\leqslant\beta_c$; on crossing it, both the bulk density and the first derivative of the current undergo finite jumps.

In the continuous-time case the phase structure is the same. The only differences are that α and β are rates which can take arbitrary non-negative values and $\alpha_c=\beta_c=1/2$. However, it is convenient to consider α and β again in the unit interval, which allows for the physical interpretation of the boundary conditions as coupling of the chain to two reservoirs: left-hand and right-hand ones with particle densities α and $1-\beta$, respectively.

The analysis of the matrix product representations for the $\rightarrow$, $\leftarrow$, and s-$\parallel$ updates reveals that the corresponding stationary states may be regarded as physically equivalent [47]. We quote the equality of the finite-size currents J_L,

$$J_L^{\rightarrow}=J_L^{\leftarrow}=J_L^{\text{s-}\parallel}, \quad (3)$$

and the relationships between the local densities $\rho_L(i)=\langle\tau_i\rangle$ at site i [47],

$$\rho_L^{\rightarrow}(i)=\rho_L^{\leftarrow}(i)-J_L^{\rightarrow},\quad \rho_L^{\text{s-}\parallel}(i)=\begin{cases}\rho_L^{\rightarrow}(i), & i \text{ odd},\\ \rho_L^{\leftarrow}(i), & i \text{ even}.\end{cases} \quad (4)$$

As shown in [48], the current $J_L^{\parallel}$ and local density $\rho_L^{\parallel}(i)$ for the TASEP with fully parallel update can also be expressed in terms of those for the forward-ordered sequential update:

$$J_L^{\parallel}=\frac{J_L^{\rightarrow}}{1+J_L^{\rightarrow}},\quad \rho_L^{\parallel}(i)=\frac{\rho_L^{\rightarrow}(i)+J_L^{\rightarrow}}{1+J_L^{\rightarrow}}. \quad (5)$$

As far as the relationships between the continuous-time and discrete-time updates are concerned, we note that the representation of the matrices $D^{\rightarrow}(\alpha,\beta;p)$ and $E^{\rightarrow}(\alpha,\beta;p)$ found in [45] is related to the representation $D_3(\alpha,\beta)$ and $E_3(\alpha,\beta)$ of the random-sequential (DEHP) algebra [42] by the limiting procedure

$$\lim_{p\to 0} pD^{\rightarrow}(p\alpha,p\beta;p)=D_3(\alpha,\beta),$$

$$\lim_{p\to 0} pE^{\rightarrow}(p\alpha,p\beta;p)=E_3(\alpha,\beta). \quad (6)$$

Therefore, the finite-size current J_L^{ct} and local density $\rho_L^{ct}(i)$ of the continuous-time TASEP follow from the corresponding quantities of the discrete-time TASEP with forward-ordered sequential dynamics in the limits

$$J_L^{ct}(\alpha,\beta)=\lim_{p\to 0} J_L^{\rightarrow}(p\alpha,p\beta;p)/p,$$

$$\rho_L^{ct}(i|\alpha,\beta)=\lim_{p\to 0}\rho_L^{\rightarrow}(i|p\alpha,p\beta;p). \quad (7)$$

The relationship between the parallel and continuous-time TASEP at algebraic level is not so straightforward, since the former case was solved either by using a quartic algebra [48] or by a bond-oriented matrix-product ansatz [49]. However, from the definition of the parallel dynamics it is clear that by rescaling the injection and removal probabilities, $\alpha=p\tilde{\alpha}$, $\beta=p\tilde{\beta}$, and taking the limit $p\to 0$ one will recover the results for the continuous-time TASEP with injection and removal rates $\tilde{\alpha}$, $\tilde{\beta}$, respectively. At the level of the grand canonical generating functions the above limiting process was carried out in [34].

Since the relationships (3) and (4) are rather trivial, in the present study we shall confine ourselves to the cases of continuous-time (random-sequential) update and two of the discrete-time updates: forward-ordered sequential and parallel.

III. FINITE-SIZE SCALING AT THE CONTINUOUS PHASE TRANSITION

As we mentioned in the Introduction, the notions of the Privman-Fisher [50] FSS have been recently extended to nonequilibrium phase transitions in models belonging to the directed percolation and diffusion-annihilation universality classes, [51–53]. The first step in the analytic confirmation of FSS for an exactly solved model of a driven lattice gas with open boundaries was made in [11] where the TASEP with forward-ordered sequential dynamics was studied.

Let us consider first the continuous phase transition across the boundary $\alpha=\alpha_c$, $\beta_c\leqslant\beta\leqslant 1$ between region AII of the low-density phase and the maximum-current phase; see Fig. 1.

According to the basic FSS hypotheses, the FSS variable in the case of a second-order transition, characterized by diverging bulk correlation length λ, should be given by the ratio L/λ, where L is the finite size of the system. As is well known, in the case at hand the correlation length λ_α (λ_β) in the low- (high-) density phase depends on whether the update is a discrete- (superscript #) or continuous-time one (superscript ct) (see, e.g., [41,49]):

$$1/\lambda_\sigma^{\#}=\ln\left[1+\frac{(\sigma_c-\sigma)^2}{\sigma(p-\sigma)}\right],$$

$$1/\lambda_\sigma^{ct}=\ln\left[1+\frac{(1/2-\sigma)^2}{\sigma(1-\sigma)}\right]\quad(\sigma=\alpha,\beta), \tag{8}$$

and $1/\lambda\equiv 0$ in the maximum-current phase. The relationship between the discrete-time and continuous-time correlation lengths is $\lambda_\sigma^{ct}=\lim_{p\to 0}\lambda_{p\sigma}^{\#}$ $(\sigma=\alpha,\beta)$. Naturally, since we study a boundary-induced phase transition, the physical quantity which measures the distance to the steady-state critical point is the deviation of the injection probability (or rate) from its critical value. From Eq. (8) we deduce that the critical exponent of the correlation length for the second-order phase transition is $\nu=2$, irrespectively of the type of dynamics.

In agreement with the equilibrium theory, the FSS variable as $L\to\infty$ and $\alpha\to\alpha_c^-$ is expected to be given by [11]

$$L/\lambda^{\#}\simeq\frac{L(\alpha_c-\alpha)^2}{\alpha_c(p-\alpha_c)}:=(x_1^{\#})^2, \tag{9}$$

in the case of a discrete-time update, and by

$$L/\lambda^{ct}\simeq 4L(1/2-\alpha)^2:=(x_1^{ct})^2, \tag{10}$$

in the case of a continuous-time update.

The exact finite-chain results obtained in [11,45,49] are conveniently expressed in terms of the parameters

$$d=\sqrt{1-p},\quad a=d+d^{-1},\quad \xi=\frac{p-\alpha}{\alpha d},\quad \eta=\frac{p-\beta}{\beta d}, \tag{11}$$

which will be used here for the discrete-time updates. The corresponding parameters for the continuous-time dynamics are

$$\tilde{d}=1,\quad \tilde{a}=2,\quad \tilde{\xi}=\frac{1-\alpha}{\alpha},\quad \tilde{\eta}=\frac{1-\beta}{\beta}. \tag{12}$$

In terms of the above notation we specify the sign of the finite-size scaling variables defined in Eqs. (9) and (10) as follows:

$$x_1^{\#}=(a+2)^{-1/2}L^{1/2}(\xi-1),\quad x_1^{ct}=(1/2)L^{1/2}(\tilde{\xi}-1). \tag{13}$$

In Ref. [11] the finite-size current $J_L^{\rightarrow}=Z_{L-1}^{\rightarrow}/Z_L^{\rightarrow}$ was analyzed by using the following exact representation of $Z_L^{\rightarrow}$ in the subregion AII of the low-density phase $(\alpha<\alpha_c<\beta)$ [45]:

$$Z_L^{\mathrm{AII},\rightarrow}=\left(\frac{d}{p}\right)^L\frac{\xi-\xi^{-1}}{\xi-\eta}(a+\xi+\xi^{-1})^L+Z_L^{\mathrm{C},\rightarrow}. \tag{14}$$

Here the expression for the normalization factor in the maximum-current phase (region C of the phase diagram) when $\alpha\neq\beta$ is

$$Z_L^{\mathrm{C},\rightarrow}=\left(\frac{d}{p}\right)^L\frac{\xi I_L(\xi)-\eta I_L(\eta)}{\xi-\eta}, \tag{15}$$

where the integral

$$I_L(\xi)=\frac{2}{\pi}\int_0^\pi \mathrm{d}\phi\frac{(a+2\cos\phi)^L\sin^2\phi}{1-2\xi\cos\phi+\xi^2} \tag{16}$$

is a nonanalytic function of ξ at $\xi=1$. For all finite L the normalization factor $Z_L^{\mathrm{AII},\rightarrow}$ in region AII represents an analytic continuation of $Z_L^{\mathrm{C},\rightarrow}$ from the domain $\alpha>\alpha_c$ to the domain $\alpha<\alpha_c$; see [45].

A direct application of the Laplace method for evaluation of the integral (16) as $L\to\infty$ shows that it changes its leading-order asymptotic behavior from $O(L^{-3/2})$ for $\xi\neq 1$ to $O(L^{-1/2})$ for $\xi=1$; see Eq. (14) in [54]. The finite-size expression that interpolates between these limiting asymptotic forms can be readily obtained by using small-argument expansion of the trigonometric functions in the integrand. The result for $x_1(L,t)=O(1)<0$ is [11]

$$I_L(\xi)\simeq\frac{(a+2)^{L+1/2}}{\xi\sqrt{\pi L}}X(-x_1), \tag{17}$$

where the FSS function X is given by

$$X(x) = 1 - \sqrt{\pi}x e^{x^2}[1 - \Phi(x)], \quad \Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{18}$$

Remarkably, the function X is closely related to the FSS form of the partition function $\hat{Z}_{2L}^{(1)}(\kappa)$ for fully directed walks (actually, Dyck paths) of $2L$ steps on the diagonally rotated square lattice, which begin and end on a wall (the x axis) and have weight κ ascribed to each contact with the wall [55]. Near the binding-unbinding transition, when $\kappa \to \kappa_c = 2$ and $L \to \infty$, the following FSS behavior was found in Ref. [55]:

$$\hat{Z}_{2L}^{(1)}(\kappa) \simeq \frac{4^L}{L^{1/2}}\hat{\varphi}^{(1)}(v), \quad v = \frac{\kappa - 2}{2}L^{1/2}, \tag{19}$$

where

$$\hat{\varphi}^{(1)}(v) = \frac{2}{\sqrt{\pi}}X(-v). \tag{20}$$

The fact that the above relationship is not accidential can be understood if one realizes that the normalization factor for the continuous-time TASEP in the limit $\beta \to \infty$ becomes

$$Z_L^{ct}(\alpha,\infty) = \sum_{p=1}^{L} B_{L,p}\alpha^{-p} = \alpha\hat{Z}_{2L}^{(1)}(1/\alpha), \tag{21}$$

where the Ballot number $B_{L,p}$ gives the number of Dyck paths with length $2L$ and p returns to the x axis. The equality (21) expresses the known representation of $Z_L^{ct}(\alpha,\infty)$ as the partition function of Dyck paths for which each contact with the x axis, apart from the start, has a weight $1/\alpha$ [34]. For these paths the binding-unbinding transition takes place at injection rate $\alpha_c = 1/\kappa_c = 1/2$, and the FSS variable becomes $v \simeq 2(1/2 - \alpha)L^{1/2} = x_1^{ct}$.

To continue our analysis, we substitute Eq. (17) into Eq. (15) and keeping the $1/L$ corrections obtain the asymptotic form

$$Z_L^{C,\to} \simeq \left(\frac{d}{p}\right)^L \frac{1}{\xi - \eta}\frac{(a+2)^{L+1/2}}{\sqrt{\pi L}}\left[X(|x_1|) - \frac{\eta}{(1-\eta)^2}\frac{a+2}{2L}\right]. \tag{22}$$

The finite-size corrections to the thermodynamic current at fixed $\beta > \beta_c$ were evaluated to the leading order as $L \to \infty$. The results are [11] in region C of the phase diagram, as $\alpha \to \alpha_c^+$,

$$J_L^{C,\to} - J_\infty^{C,\to} \simeq \frac{1}{L}J_\infty^{C,\to}\left[\frac{1}{2X(|x_1|)} - x_1^2\right], \tag{23}$$

and in region AII, as $\alpha \to \alpha_c^-$,

$$J_L^{AII,\to} - J_\infty^{AII,\to} \simeq \frac{1}{L}J_\infty^{C,\to}\left[\frac{1}{4\sqrt{\pi}x_1 e^{x_1^2} + 2X(x_1)}\right]. \tag{24}$$

Here

$$J_\infty^{C,\to} \equiv J_\infty^{MC,\to} = \frac{1-(1-p)^{1/2}}{1+(1+p)^{1/2}}, \quad J_\infty^{AII,\to} \equiv J_\infty^{LD,\to} = \frac{\alpha(p-\alpha)}{p(1-\alpha)} \tag{25}$$

are the thermodynamic currents in the maximum-current and low-density phases, respectively.

As a bulk order parameter of the continuous phase transition we consider the difference $\Delta_\infty(\alpha;p) := J_\infty(\alpha=1;p) - J_\infty(\alpha;p)$. In the case of forward-ordered sequential dynamics we obtain, from Eq. (25),

$$\Delta_\infty^{\to}(\alpha;p) = \begin{cases}(\alpha-\alpha_c)^2/[p(1-\alpha)], & \alpha \leqslant \alpha_c,\\ 0, & \alpha \geqslant \alpha_c,\end{cases} \tag{26}$$

and, according to Eq. (7), in the case of continuous-time dynamics,

$$\Delta_\infty^{ct}(\alpha) = \lim_{p\to 0}\Delta_\infty^{\to}(p\alpha;p)/p = \begin{cases}(\alpha-1/2)^2, & \alpha \leqslant 1/2,\\ 0, & \alpha \geqslant 1/2.\end{cases} \tag{27}$$

The above results suggest that the critical exponent for the order parameter has the universal value of 2.

A consistent choice of the finite-size order parameter, different from the one considered in [11], is given by $\Delta_L(\alpha;p) := J_L(\alpha=1;p) - J_L(\alpha;p)$. With the aid of Eqs. (23) and (24) and the appropriate relationships between the currents corresponding to the different types of dynamics, we obtain that the leading-order FSS for the order parameter has the form

$$\Delta_L - \Delta_\infty \simeq \frac{1}{L}J_\infty^{MC}G(x_1), \tag{28}$$

where the FSS function G is universal:

$$G(x_1) = \begin{cases}\frac{3}{2} + x_1^2 - [2X(|x_1|)]^{-1}, & x_1 \leqslant 0,\\ \frac{3}{2} - [4\sqrt{\pi}x_1 e^{x_1^2} + 2X(x_1)]^{-1}, & x_1 \geqslant 0.\end{cases} \tag{29}$$

Note that the bulk order parameter Δ_∞ and the amplitude on the right-hand side of Eq. (28) depend on the particular dynamics. The explicit dependence of the FSS variable L/λ on the parameters of the problem is also nonuniversal [compare Eqs. (9) and (10)], but depends only on the fact whether the dynamics is a continuous-time or discrete-time one.

Since the current in the high-density phase maps onto the low-density phase under the exchange of arguments $\alpha \leftrightarrow \beta$, the FSS properties of the continuous transition across the boundary $\beta = \beta_c$, $\alpha_c \leqslant \alpha \leqslant 1$ between the high-density phase and the maximum-current phase follow trivially from the above results and the particle-hole symmetry.

IV. FINITE-SIZE SCALING AT THE FIRST-ORDER TRANSITION

In the thermodynamic limit the first-order phase transition occurs across the borderline $\beta = \alpha$, $0 \leqslant \alpha \leqslant \alpha_c$ $(\eta = \xi, \xi \geqslant 1)$ between subregions AI and BI of the phase diagram; see

Fig. 1. It manifests itself by a finite jump in the bulk density, $\Delta\rho_\infty=\rho_\infty^{\rm HD}-\rho_\infty^{\rm LD}>0$, where $\rho_\infty^{\rm HD}$ ($\rho_\infty^{\rm LD}$) is the bulk density in the high- (low-) density phase at the transition point. The magnitude of the jump at $\alpha=\beta$ has the general form $\Delta\rho_\infty=A(\beta;p)(\beta_c-\beta)$, where $A(\beta;p)$ is an amplitude which depends on the specific update: $A^{ct}=2$ and

$$A^{\rightarrow}(\beta;p)=\frac{1+(1-p)^{1/2}-\beta}{p(1-\beta)},$$

$$A^{\parallel}(\beta;p)=\frac{p(1-\beta)}{(p-\beta^2)}A^{\rightarrow}(\beta;p). \tag{30}$$

We remind the reader that $\beta_c=1-(1-p)^{1/2}\equiv\beta_c^{\#}$ for all the discrete-time updates and $\beta_c=1/2\equiv\beta_c^{ct}$ for the continuous-time dynamics.

It is known that the nonequilibrium first-order phase transition in the TASEP is characterised by a *diverging* correlation length λ defined as [41]

$$1/\lambda=|1/\lambda_\alpha-1/\lambda_\beta|=C_2(\beta,p)|\alpha-\beta|+O(|\alpha-\beta|^2), \tag{31}$$

where the factor $C_2(\beta,p)$ depends on whether the update is discrete or continuous time:

$$C_2^{\#}(\beta,p)=\frac{[1+(1-p)^{1/2}-\beta](\beta_c^{\#}-\beta)}{\beta(1-\beta)(p-\beta)},$$

$$C_2^{ct}(\beta)=\frac{2(1/2-\beta)}{\beta(1-\beta)}. \tag{32}$$

Here and below the superscript # stands for all the discrete-time updates. As expected, $C_2^{ct}(\beta)=\lim_{p\to 0}C_2^{\#}(p\beta,p)$. From Eq. (31) it follows that the correlation length at the first-order phase transition in the TASEP is described by a critical exponent $\nu=1$, irrespectively of the update.

In Ref. [11] the case of forward-ordered sequential update was analyzed and the finite-size scaling variable $x_2=C_2(\beta-\alpha)L$ was introduced. Note that x_2 is the natural extension to negative values of the variable $L/\lambda\simeq C_2|\alpha-\beta|L$. By using the techniques of [11] and relationships (5) and (7), one can readily show that close to the first-order transition line $\alpha=\beta<\alpha_c=\beta_c$ there exists a universal finite-size scaling function for the local density on the macroscopic scale, $i/L=r$, $0<r<1$:

$$\rho_L(rL|\beta;p)=\rho_\infty^{\rm LD}(\beta;p)+\Delta\rho_\infty(\beta;p)\frac{e^{x_2 r}-1}{e^{x_2}-1}. \tag{33}$$

Here the coefficients $\rho_\infty^{\rm LD}$ and $\Delta\rho_\infty$ depend on the specific update [see Eq. (30)] and recall that

$$\rho_\infty^{\rm LD,\rightarrow}(\alpha;p)=\frac{\alpha(1-p)}{p(1-\alpha)},\quad \rho_\infty^{\rm LD,\parallel}(\alpha;p)=\frac{\alpha(1-\alpha)}{p-\alpha^2},$$

$$\rho_\infty^{\rm LD,ct}(\alpha)=\alpha. \tag{34}$$

In the limit $x_2\to 0$ expression (33) reduces to the well-known linear density profile on the coexistence line, $\rho_L(rL)=\rho_\infty^{\rm LD}+r\Delta\rho_\infty$, and in the limit $x_2\to+\infty$ ($x_2\to-\infty$) one recovers the bulk density in the low-density (high-density) phase.

V. ZEROS OF THE NORMALIZATION FACTOR IN THE COMPLEX PLANE

As was mentioned in the Introduction, there is a sound evidence that the zeros of the steady-state probability normalization factor Z_L in Eq. (1)—say, in the plane of a complex injection probability (rate) $\alpha=x+iy$—may provide useful information about the nonequilibrium phase transitions of the TASEP. Evidently, Z_L is defined up to a factor which may depend on the parameters of the problem (e.g., on p, α, β). Indeed, if one multiplies the steady-state weights of all the configurations by the same factor, the probability distribution will not change but that common factor will appear in the normalization Z_L. For example, the explicit result for the normalization Z_L^{ct} in the continuous-time TASEP given by Eq. (B10) in Ref. [42], denoted here with the subscript DEHP, and the one given by Eq. (6.1) at $q=0$ in Ref. [56], with the subscript USW, differ by a rate-dependent factor:

$$Z_L^{ct}|_{\rm DEHP}=\frac{\alpha+\beta-1}{\alpha\beta}\,Z_L^{ct}|_{\rm USW}. \tag{35}$$

Obviously, the additional zero of the expression on the left-hand side at the mean-field line $\alpha+\beta=1$ is irrelevant to the phase transitions. In our numerical study we will use the expression from Ref. [42]. Similarly, the result for $Z_L^{\parallel}$ in Ref. [49], denoted here with the subscript GN, differs from the one in Ref. [48], subscript ERS, by a factor depending on all the transition probabilities:

$$Z_L^{\parallel}|_{\rm GN}=\frac{(\alpha\beta)^L}{p}\,Z_L^{\parallel}|_{\rm ERS}. \tag{36}$$

In our study we will use for $Z_L^{\parallel}$ the result of Ref. [48] which is rather simply related to our result [45] for $Z_L^{\rightarrow}$:

$$Z_L^{\parallel}=p^L Z_L^{\rightarrow}[1+J_L^{\rightarrow}]. \tag{37}$$

One can readily show that at fixed $0<\beta\leqslant 1$ the zeros of the factor $[1+J_L^{\rightarrow}]$ are at the points $\alpha=\pm p^{1/2}$ which lie out of the domain of the low-density phase, since $0<\alpha<1-(1-p)^{1/2}<p^{1/2}$ for all $0<p<1$. Hence, the zeros of $Z_L^{\parallel}$ which are relevant to the phase transition coincide with those of $Z_L^{\rightarrow}$.

Note that due to Eqs. (6) and (37) the normalization factors for the continuous-time and discrete-time dynamics are related by the following limits:

$$Z_L^{ct}(\alpha,\beta)=\lim_{p\to 0}Z_L^{\parallel}(p\alpha,p\beta;p)=\lim_{p\to 0}p^L Z_L^{\rightarrow}(p\alpha,p\beta;p). \tag{38}$$

In deriving the last equality we have taken into account that $J_L^{\rightarrow}(p\alpha,p\beta;p)\to 0$ as $p\to 0$; see Eq. (7).

The loci of the zeros of the normalization factor in the thermodynamic limit can easily be obtained with the aid of the function

$$g(z) := -\lim_{L\to\infty}\frac{1}{L}\ln Z_L(z,\beta;p), \tag{39}$$

which is the analog of the free energy density. Note that in the cases of continuous-time [29,30] and forward-ordered sequential dynamics, the normalization behaves for large L as $Z_L \simeq AL^{\gamma}J_{\infty}^{-L}$, where the current J_∞ in the thermodynamic limit, the amplitude A, and the exponent γ depend on the update and the phase under consideration. Thus we obtain

$$g^{ct}(z) = \ln J_\infty^{ct}(z,\beta), \quad g^{\rightarrow}(z) = \ln J_\infty^{\rightarrow}(z,\beta;p), \tag{40}$$

which is not the case for the parallel update, when from Eq. (37) it follows that

$$g^{\parallel}(z) = \ln J_\infty^{\rightarrow}(z,\beta;p) - \ln(p). \tag{41}$$

The phase boundary in the complex plane $z=\mathrm{Re}(\alpha)+\mathrm{i}\,\mathrm{Im}(\alpha)$ is defined by the equation [29] $\mathrm{Re}\, g_1(z)=\mathrm{Re}\, g_2(z)$ or, equivalently, by

$$|\ln J_1(z,\beta;p)| = |\ln J_2(z,\beta;p)|. \tag{42}$$

Here the subscripts 1 and 2 refer to two different phases, so that there are points z in the complex plane such that $g_1(z)\neq g_2(z)$.

Below we compare the available analytical and numerical results for the distribution of the zeros of the probability normalization factor $Z_L(z)$ in the cases of continuous-time and forward-ordered sequential dynamics both in the thermodynamic limit and in the finite-size case.

A. Thermodynamic limit

The results for the TASEP with continuous-time dynamics were obtained and analyzed by Blythe and Evans [29,30]. Here we reproduce some of them for the sake of comparison with the discrete-time updates. By taking into account the expressions for the current in the different phases, $J_\infty^{\mathrm{LD},ct}=\alpha(1-\alpha)$, $J_\infty^{\mathrm{HD},ct}=\beta(1-\beta)$, and $J_\infty^{\mathrm{MC},ct}=1/4$, one obtains in the thermodynamic limit exact analytical results.

(a) The line of zeros in the plane of complex $\alpha=x+\mathrm{i}y$ at the second-order phase transition (at $\alpha=1/2$, $\beta>1/2$) between the low-density and maximum-current phases is given by the equation

$$y = \pm\{-1/4-(1/2-x)^2+[1/16+(1/2-x)^2]^{1/2}\}^{1/2}. \tag{43}$$

As shown in [29], close to the transition point $\alpha=1/2$ the asymptotic form of the locus of zeros has two branches $y=\pm(1/2-x)$, $x\leqslant 1/2$, which cross the positive real axis at impact angle of $\pi/4$; see the solid curve at $\beta=1$ in Fig. 2. For the functions $g^{ct}(z)$ in the corresponding two phases close to the critical point $z_c=1/2$ one has

$$g^{\mathrm{LD},ct}(z) \simeq \ln(1/4) - 4(z-z_c)^2, \quad g^{\mathrm{MC},ct}(z) = \ln(1/4). \tag{44}$$

By using Eq. (19) of Ref. [30] one obtains that the density of zeros $\mu(s)$ at the positive real axis, as a function of the arch length s of the phase boundary, decreases to zero linearly with $s\to 0$ as

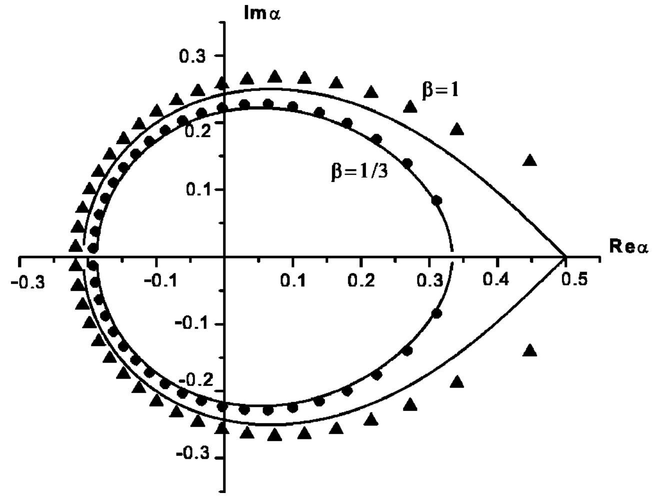

FIG. 2. Distribution of the zeros of Z_L^{ct} in the complex-α plane at $L=40$: the solid triangles are for $\beta=1$, and the solid circles for $\beta=1/3$. The solid curves show the corresponding lines of zeros in the limit $L\to\infty$.

$$\mu^{ct}(s) = 4s/\pi. \tag{45}$$

This result is expected for a continuous phase transition in the equilibrium Lee-Yang theory.

(b) The line of zeros in the plane of $\alpha=x+\mathrm{i}y$ at the first-order phase transition (at $\alpha=\beta<1/2$) between the low-density and high-density phases is given by the equation

$$y = \pm\{-1/2+x-x^2+[(1/2-x+x^2)^2 + \beta^2(1-\beta)^2-x^2(1-x)^2]^{1/2}\}^{1/2}. \tag{46}$$

As shown in [29], close to the transition point the asymptoic form of the locus of zeros is a smooth curve that crosses the positive real axis at right angle; see the solid curve at $\beta=1/3$ in Fig. 2. For the functions $g^{ct}(z)$ in the low- and high-density phases close to $z_c=\beta$ one has

$$g^{\mathrm{LD},ct}(z) \simeq \ln\beta(1-\beta) - \frac{1-2\beta}{\beta(1-\beta)}(z-z_c),$$

$$g^{\mathrm{HD},ct}(z) = \ln\beta(1-\beta). \tag{47}$$

In view of Eq. (18) of Ref. [30], the above expressions imply that at the first-order phase transition the density of zeros $\mu(0)$ at the positive real axis is constant,

$$\mu^{ct}(0;\beta) = \frac{1-2\beta}{2\pi\beta(1-\beta)}, \tag{48}$$

in agreement with the equilibrium Lee-Yang theory.

In the case of forward-ordered sequential update, by expanding the function $g^{\rightarrow}(z)$ [see Eq. (40)] in the neighborhood of the continuous phase transition point $z_c=\alpha_c$ (at fixed $\beta>\beta_c$), we obtain

$$g^{\mathrm{LD},\rightarrow}(z) \simeq \ln J_\infty^{\mathrm{MC},\rightarrow} - \frac{(z-z_c)^2}{(1-p)^{1/2}[1-(1-p)^{1/2}]^2},$$

$$g^{\mathrm{MC},\rightarrow}(z) = \ln J_\infty^{MC,\rightarrow}. \tag{49}$$

Hence, the density of zeros $\mu^{\rightarrow}(s;p)$ at the positive real axis, as a function of the arch length s of the phase boundary again decreases to zero linearly with $s \to 0$,

$$\mu^{\rightarrow}(s;p) = \frac{s}{\pi(1-p)^{1/2}[1-(1-p)^{1/2}]^2}, \tag{50}$$

but the proportionality coefficient differs from the one for the continuous-time dynamics; see Eq. (45). In view of Eq. (38), the definition (39) yields the relationship

$$g^{ct}(z;\beta) = \lim_{p\to 0}[g^{\rightarrow}(pz,p\beta;p) - \ln(p)], \tag{51}$$

and hence $\mu^{ct}(s) = \lim_{p\to 0}[p^2\mu^{\rightarrow}(s;p)]$.

In the neighborhood of the first-order phase transition at $z_c = \beta < \beta_c^{\#}$ we obtain

$$g^{\mathrm{LD},\rightarrow}(z) \simeq \ln\frac{\beta(p-\beta)}{p(1-\beta)} - \frac{\beta^2-2\beta+p}{\beta(1-\beta)(p-\beta)}(z-z_c),$$

$$g^{\mathrm{HD},\rightarrow}(z) = \ln\frac{\beta(p-\beta)}{p(1-\beta)}. \tag{52}$$

Again the density of zeros $\mu^{\rightarrow}(0;\beta,p)$ at the positive real axis equals a constant,

$$\mu^{\rightarrow}(0;\beta,p) = \frac{\beta^2-2\beta+p}{2\pi\beta(1-\beta)(p-\beta)}, \tag{53}$$

different from the one for the continuous-time dynamics. As follows from Eq. (51), $\mu^{ct}(0;\beta) = \lim_{p\to 0}[p\mu^{\rightarrow}(0;p\beta,p)]$.

In the thermodynamic limit the line of zeros of the normalization factor $Z_L^{\rightarrow}(z)$ is given by the equation

$$y = \pm\{-(p^2-c^2)/2 + px - x^2 + \sqrt{[(p^2-c^2)/2 - px + x^2]^2 + c^2(1-x)^2 - x^2(p-x)^2}\}^{1/2}. \tag{54}$$

Here

$$c = pJ_\infty^{\mathrm{MC},\rightarrow} = [1-(1-p)^{1/2}]^2 \tag{55}$$

for the second-order phase transition and

$$c = pJ_\infty^{\mathrm{HD},\rightarrow} = \beta(p-\beta)/(1-\beta) \tag{56}$$

for the first-order phase transition. These curves are illustrated in Fig. 3 for the particular values of $p=3/4$ and $\beta=3/4$ or $\beta=1/3$, respectively. They have the same general shape and asymptotic behavior close to the positive real axis as the corresponding ones in the case of continuous-time dynamics. Moreover, under the replacement $\alpha \to p\alpha$ (hence, $x \to px$, $y \to py$) and $\beta \to p\beta$, in the limit $p \to 0$ the curve (54) with substitution (55) maps onto the curve (43) and with the substitution (56) onto the curve (46).

B. Finite-size behavior

The loci of the zeros of $Z_L(z)$ in the complex-α plane for a finite chain of $L=40$ sites are shown in Fig. 2 for continuous-time dynamics and in Fig. 3 for forward-ordered sequential dynamics with $p=3/4$, so that $\alpha_c=1/2$ in both cases. The values of β are chosen so that one of them corresponds to a second-order phase transition ($\beta=1$ in the former case and $\beta=3/4$ in the latter one) and the other one to a first-order transition ($\beta=1/3$ in both cases). The similarity of the patterns in the cases of continuous-time and discrete-time dynamics is obvious; see also Figs. 3 and 4 in Ref. [35] for the parallel-update TASEP.

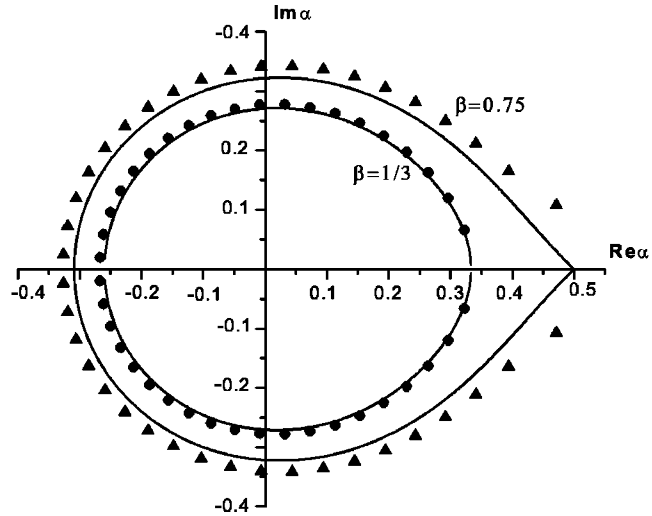

FIG. 3. Distribution of the zeros of $Z_L^{\rightarrow}$ in the complex-α plane at $L=40$ and $p=3/4$: the solid triangles are for $\beta=3/4$ and the solid circles for $\beta=1/3$. The solid curves show the corresponding lines of zeros in the limit $L\to\infty$.

To derive quantitative information, we consider the closest to the positive real axis pair $\{z_L, \overline{z_L}\}$ of complex-conjugate zeros of $Z_L(z)$, $z = \mathrm{Re}(\alpha) + \mathrm{i}\,\mathrm{Im}(\alpha)$, and evaluate the rate of decrease of the distance $|z_L - \alpha_c|$ as $L \to \infty$ for both the continuous-time and forward-ordered sequential dynamics.

In the case of a nonequilibrium second-order phase transition the best linear fits to the corresponding log-log plots for several chain sizes L are shown in Fig. 4. The results clearly indicate a power-law asymptotic behavior of the form

$$|z_L - \alpha_c| \simeq AL^{-\theta}, \tag{57}$$

with amplitude $A \simeq 1.031(8)$ and shift exponent $\theta \simeq 0.525(1)$ in the case of continuous-time dynamics, $A \simeq 0.784(8)$ and $\theta \simeq 0.536(3)$ in the case of forward-ordered sequential dynamics. Note that the estimated values of θ are close to $1/\nu = 1/2$, where $\nu=2$ is the critical exponent of the bulk correlation length for the nonequilibrium second-order phase transition. Remarkably, the same value $1/2$ of the shift exponent θ was found by Bena *et al.* [31] for their urn model which also exhibits a nonequilibrium second-order phase transition in the thermodynamic limit. Our estimates of the amplitudes for the different type of dynamics differ considerably.

The case of a nonequilibrium first-order phase transition was studied for the particular value of $\beta=1/3$ (and $p=3/4$ in the case of discrete-time dynamics) when the transition point is at $\alpha_c=1/3$. Our best linear fits to the log-log plots of the

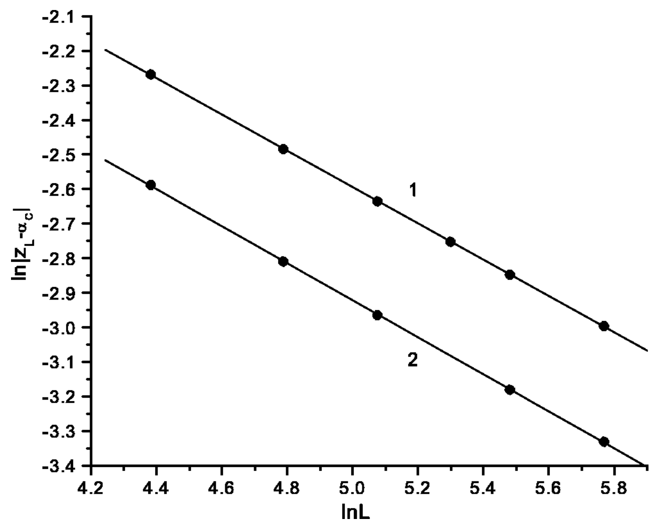

FIG. 4. A log-log plot of the shortest distance between the set of zeros of Z_L in the complex-α plane and the second-order phase transition point $\alpha_c=1/2$ for several chain sizes L. The straight lines show the best linear fits to our data obtained for the TASEP with continuous-time (line 1) and forward-ordered sequential (line 2) dynamics.

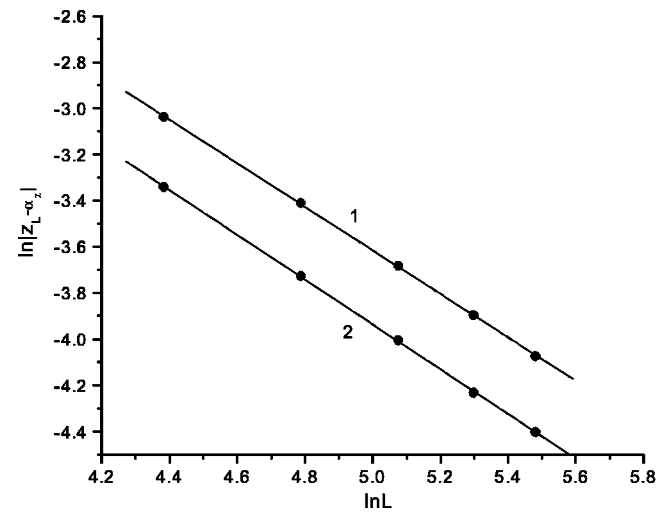

FIG. 5. A log-log plot of the shortest distance between the set of zeros of Z_L in the complex-α plane and the first-order phase transition point $\alpha_c=\beta=1/3$, for several chain sizes L. The straight lines show the best linear fits to our data obtained for the TASEP with continuous-time (line 1) and forward-ordered sequential (line 2) dynamics.

distance $|z_L-\alpha_c|$ as a function of the chain size L are shown in Fig. 5. They strongly suggest a power-law asymptotic dependence of the form (57) with estimated values: $A \simeq 3.002(9)$ and $\theta \simeq 0.943(6)$ for the continuous-time dynamics, $A \simeq 2.489(8)$ and $\theta \simeq 0.970(7)$ for the forward-ordered sequential dynamics. The above values of θ are close to $1/\nu=1$, where $\nu=1$ is the critical exponent of the bulk correlation length for the nonequilibrium first-order phase transition. The amplitudes obtained for the considered updates are definitely different.

VI. CONCLUSIONS

Our results support the conclusion that the versions of the TASEP, based on different update rules, belong to the same *nonequilibrium* FSS universality class. The critical exponent of the correlation length is $\nu=2$ for the second-order phase transition and $\nu=1$ for the first-order one. The finite-size scaling functions are of the same shape for each transition order, but differ by nonuniversal prefactors which depend on the specific update. The explicit dependence of the FSS variable L/λ on the parameters of the model (the probabilities α, β, p or the rates α, β) changes on passing from continuous-time to a discrete-time dynamics.

From our numerical results on the rate of covergence of the zeros z_L, $\overline{z_L}$ of the normalization factor $Z_L(z)$ in the plane of complex injection probability (or rate) to the transition point $\alpha=\alpha_c>0$ of the infinite chain, we conjecture that the FSS prediction

$$|z_L-\alpha_c| = |\overline{z_L}-\alpha_c| \simeq AL^{-1/\nu} \tag{58}$$

is obeyed with a universal shift exponent $1/\nu$. On the grounds of the above argument and the plausible assumption that the zeros which are closest to the positive real axis dominate the analytical properties of $Z_L(z)$ for the physically meaningful positive values of z, one can regard z_L (or $\overline{z_L}$) as a shifted pseudocritical point. The amplitude A in the above finite-size shift relationship appears to be nonuniversal.

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